MATH2050C Assignment 2

Deadline: Jan 24, 2018.

Hand in: Section 2.2 no. 7, 10(b), 15(a)(d). Section 2.3 no. 7.

Section 2.2 no. 6, 7, 10, 11, 12, 14, 15.

Section 2.3 no. 3, 4, 5, 6, 7.

Supplementary Exercises

1. (a) Show that $(\mathbb{Z}_5, +, \cdot)$ (see Ex 1 for definition) does not have an order like $\mathbb{Q}$ or $\mathbb{R}$.
   (b) Explain why $\mathbb{Q}$ does not have the order-completeness property. How about $\mathbb{Q}(\sqrt{2})$?
2. Given $\varepsilon > 0$, show that there is some natural number n satisfying

$$\frac{1}{n} + \frac{a}{n^2} + \frac{b}{n^2} < \varepsilon,$$

   where a, b are any real numbers.

The Real Number System: Characterization

The real number system $\mathbb{R}$ satisfies the following three properties: (1) $(\mathbb{R}, +, \cdot)$ forms a field (see the notes in Ex 1 or Text), (2) it has an order (any two real numbers can be compared) induced by the existence of a subset $\mathbb{P}$ (the set of positive numbers), see Text for details, and (3) order-completeness property: *Every non-empty subset of* $\mathbb{R}$ *bounded from above has a supremum in* $\mathbb{R}$.

Consider the following mathematical structures: (a) $\mathbb{Z}_5$ (see Supp Ex 1), (b) $\mathbb{Q}$, (c) $\mathbb{Q}(\sqrt{2})$ (see Ex 9 in 2.1 in Text) and (d) $\mathbb{R}$. You may check that (a)-(d) all satisfy (1), (b)-(d) satisfy (2) and (3), and only (d) satisfies (1)-(3).

(1), (2) and (3) characterizes the real number system in the sense that in case any other mathematical structure satisfying (1), (2), and (3) is the same as $\mathbb{R}$ after re-labeling. In mathematical terms, it is called an order-preserving isomorphism. Google for more.

Our Text presumes the existence of the real number system and use (1), (2) and (3) to characterize it. It says nothing about the *existence* of a mathematical structure enjoying these three properties. Indeed, the construction of the real number is known. The idea is first to construct the natural number system by the so-called Peano's axioms and then use them to construct integers and rational numbers. To construct real numbers from rational numbers, a sophisticated step is needed. There are two common approaches, one by Dedekind cuts and the other by Cantor sequences. We will not go into this, Google for more if you interested.